N90-23074

MODEL REDUCTION IN THE PHYSICAL COORDINATE SYSTEM

K. Harold Yae and K.Y. Jeong
Center for Simulation and Design Optimization
Department of Mechanical Engineering
The University of Iowa
Iowa City, Iowa 52242
(319) 335-5683

ABSTRACT

In the dynamics modeling of a flexible structure, finite element analysis employs reduction techniques, such as Guyan's reduction, to remove some of the "insignificant" physical coordinates, thus producing a dynamics model that has smaller mass and stiffness matrices. But this reduction is limited in the sense that it removes certain degrees of freedom at a node point, instead of node points themselves in the model. From the standpoint of linear control design, the resultant model is still too large despite the reduction. Thus, some form of model reduction is frequently used in the control design by approximating a large dynamical system with a fewer number of state variables. However, a problem arises from the placement of sensors and actuators in the reduced model, because a model usually undergoes, before being reduced, some form of coordinate transformations that do not preserve the physical meanings of the states. To correct such a problem, a method is developed that expresses a reduced model in terms of a subset of the original states.

The proposed method starts with a dynamic model that is originated and reduced in finite element analysis. Then the model is converted to the state space form, and reduced again by the internal balancing method. At this point, being in the balanced coordinate system, the states in the reduced model have no apparent resemblance to those of the original model. Through another coordinate transformation that is developed in this paper, however, this reduced model is expressed by a subset of the original states.

INTRODUCTION

In the dynamics modeling of a structure, finite element analysis employs reduction techniques, such as Guyan's reduction, to remove some of the "insignificant" physical coordinates [6, Guyan 1965; 10, Irons 1965], thereby producing a model that has smaller mass and stiffness matrices. But this reduction is limited in the sense that it reduces degrees of freedom at a node point, instead of the number of node points in the model. From the standpoint of linear control design, the resultant model is still too large despite the reduction, because the size of a model depends on degrees of freedom at each node and the number of node points.

In the control literature, there has been extensive research and publication on model reduction methods [5, Genesio and Milanese 1976; 7, Hickin and Sinha 1980], in which the primary objective is the approximation of a large dynamical system by fewer state variables with minimal change on the input-output characteristics. For example, the aggregation method [1, Aoki 1968] reduces a model by "aggregating" the original state vector into a lower dimensional vector, in which the concept of aggregation is a generalization of that of projection and related to that of state vector partitioning. Skelton and Hughes [15, 1980] derived modal cost analysis for linear matrix second order systems that are expressed in the state space form. The decomposition of quadratic cost index into the sum of contributions from each modal coordinate is used to rank the importance of the structure's modes. The internal balancing method [12, Moore 1981; 13, Pernebo and Silverman 1982; 14, Shokoohi, Silverman, and Van Dooren 1983; 4, Gawronski and Natke 1986; 3, Gawronski and Natke 1987] is based on "measures" of controllability and observability, which are defined by the controllability and observability grammians in certain subspaces of the original state space. Then the most controllable and observable part is used as a low-order approximation for the model. Hyland and Bernstein [8, 1985] have derived the first order conditions for quadratically optimal reduced order modeling of linear time invariant systems, in which they show how the complex optimality conditions in [16, Wilson 1970] can be transformed, without loss of generality, into much simpler and more tractable forms. The transformation is facilitated by exploiting the presence of an oblique (i.e., nonorthogonal) projection that was not recognized in [16, Wilson 1970] and that arises as a direct consequence of optimality.

From a close examination of the various reduction methods employed by the two distinctly different communities, it follows that a finite element dynamic model can be further reduced by the reduction methods used in the control community, provided that it is first converted into the state space form by assigning two states—displacement and velocity—to each degree of freedom at the node. However, a problem arises from subsequent structural control design, especially from the placement of sensors and actuators in the reduced model, because a model usually undergoes, before being reduced, some form of coordinate transformation through which a reduced model usually results in a subspace quite different from the original state space. Consequently, it is difficult, sometimes impossible, to recognize any connection between the states of the reduced model and those of the original model.

In the internal balancing method, we discovered that with an additional coordinate transformation it is possible to express the reduced model in terms of a subset of the original states. The method described in this paper proceeds with a finite element model of a structure that was already reduced by Guyan's reduction [6, Guyan 1965]. The model is then converted to the state space form, and is reduced again by the internal balancing method. At this point, being in the balanced coordinate system, the states of the reduced model have no apparent resemblance to those of the original model. But, through another coordinate transformation derived from the states that are deleted during reduction, this reduced model is expressed by a subset of the original states.

The procedure is illustrated through two examples. The first example is hypothetical, simple, yet quite effective for demonstration. The second example starts with a finite element model, and finally arrives at the reduced model that has a fewer number of node points. Throughout the two examples the impulse responses of several states are compared in the time domain.

MODEL REDUCTION BY THE INTERNAL BALANCING METHOD

The structural dynamic model in this paper is assumed to result from finite element analysis and have the following form:

$$\mathbf{M\ddot{q} + D\dot{q} + Kq = f} \tag{1}$$

where M, D and K are the nxn real, symmetric, positive definite matrices reflecting the mass, damping and stiffness properties. The nx1 vector $\mathbf{q}$ is the displacement vector; that is, each element describes the position of a node. The overdots denote differentiation with respect to time. The nx1 vector $\mathbf{f}$ represents the external forces applied to the structure. In addition, the system is assumed to be asymptotically stable (hence, the definiteness requirement on M, D and K).

Eq. (1) is first converted into the state space form such that

$$\mathbf{\dot{x} = Ax + Bu} \tag{2}$$

where the state vector **x** and the state matrix A are defined by

$$\mathbf{x} = \begin{bmatrix} \dot{q} \\ q \end{bmatrix} \quad A = \begin{bmatrix} -M^{-1}D & -M^{-1}K \\ I & 0 \end{bmatrix} \tag{3}$$

Here, I denotes the nxn identity matrix, 0 the nxn matrix of zeros, and M^{-1} the inverse of the nonsingular mass matrix M. The matrix B is called the input matrix and has a form determined by the location of the applied forces **f**. The vector **u**, often called the control force, has the form:

$$\mathbf{u} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{4}$$

From the conversion, the dimension of **u** has now become 2nx1, and the dimension of A 2nx2n. If **u** is a scalar, i.e., if the input has the same time history at each node, then B becomes a 2nx1 vector that determines the location and the magnitude of an actuator. At this point it is necessary to indicate which states are to be measured or monitored by selecting an output matrix C such that

$$y(t) = C\mathbf{x}(t) \tag{5}$$

where y(t) is a vector consisting of those states that are to be measured. Since B and C are directly related to the locations of measurement and applied force, they influence the degree of controllability and observability of the system.

The system defined by A, the choice of outputs defined by C, and the location of applied forces defined by B, must all be such that the rank of

$$U_c = \begin{bmatrix} B & AB & \cdots & A^{2n-1}B \end{bmatrix} \tag{6}$$

and

$$U_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{2n-1} \end{bmatrix} \tag{7}$$

are 2n. That is, the system defined by (A,B,C) must be both controllable and observable. For most structural models in which each part is physically connected with another, the system is controllable and observable for any single applied force and any single state measurement (see, for instance, [9, Inman 1989]).

The concepts of controllability and observability are essential to the balanced model reduction. First, each state is examined on its degree of observability--the amount of contribution by each state to the measurement of the system response, and is also examined on its degree of controllability--the effect of applied force on the system response. The balanced reduction method then suggests that the states that do not affect the response significantly be removed from the model, producing the desired reduced order model. In this way, the method attempts to find a model of the smallest size that best captures the dynamics of the structure.

The controllability and observability grammians [12, Moore 1981; 13, Pernebo and Silverman 1982], which are varying under coordinate transformations, are used to define the "measures" of controllability and observability in a certain state space. Moore [12, 1981] has shown that there exists a coordinate system in which the two grammians are equal and diagonal. The corresponding system representation is called *balanced.* The numerical algorithms of how to obtain the transformation matrix are given both by [12, Moore 1981] and by [11, Laub 1980]. In the remainder of this section the internal balancing method is briefly summarized for completeness.

The *controllability grammian*, denoted by W_c, and the *observability grammian*, denoted by W_o, are defined as:

$$W_c = \int_0^{\infty} e^{At} BB^T e^{A^T t} dt \qquad (8)$$

and

$$W_o = \int_0^{\infty} e^{A^T t} C^T C e^{At} dt \qquad (9)$$

where e^{At} is the matrix exponential function defining the state transition matrix of the system. These grammians provide a measure of how controllable and how observable a structure is with the given input and output configuration. And their values are dependent on the coordinate system in which they are evaluated.

If we denote by P the transformation of the system into the balanced coordinate system, and if we denote by $W_o(P)$ and $W_c(P)$ the grammians defined in the balanced coordinate system, then the balanced system is defined by $\hat{A} = P^{-1}AP$, $\hat{B} = P^{-1}B$, $\hat{C} = CP$, and $\hat{\mathbf{x}} = P^{-1}\mathbf{x}$. In addition, the two grammians are equal:

$$W_c(P) = W_o(P) = \mathrm{diag}\{\sigma_1, \sigma_2 \dots \sigma_{2n}\} \qquad (10)$$

where the σ_i denotes the singular values of $W_c(P)$. By arranging the singular values in descending order and permuting the states correspondingly, the states in $\hat{\mathbf{x}}$ are arranged according to their level of controllability and observability; in other words, σ_1 being the largest, $\hat{x}_1$ is the most controllable and most observable state.

The method first partitions the state, input and measurement matrices on the basis of the magnitude of the singular values. For some index 2n-k, σ_{2n-k} would be much smaller than the preceding singular value σ_{2n-k-1}. Thus the vector $\hat{\mathbf{x}}$ can be partitioned as

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_r \\ \hat{\mathbf{x}}_d \end{bmatrix} \qquad (11)$$

where $\hat{\mathbf{x}}_r$ contains (2n-k) states that are to be retained in the reduced model, and $\hat{\mathbf{x}}_d$ contains k states to be discarded in the model reduction because they correspond to small values of σ_i. These discarded coordinates are least controllable and observable; that is, they have least effect on the response of the system. Accordingly, the balanced system is partitioned as

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}_r \\ \dot{\hat{\mathbf{x}}}_d \end{bmatrix} = \begin{bmatrix} \hat{A}_r & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_r \\ \hat{\mathbf{x}}_d \end{bmatrix} + \begin{bmatrix} \hat{B}_r \\ \hat{B}_d \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \hat{C}_r & \hat{C}_d \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_r \\ \hat{\mathbf{x}}_d \end{bmatrix} \qquad (14)$$

where $\hat{A}_r$ is a (2n-k)×(2n-k) matrix representing the reduced model in the balanced coordinate system. The reduced model $(\hat{A}_r, \hat{B}_r, \hat{C}_r, \hat{\mathbf{x}}_r)$ of order (2n-k) thus results from the balanced representation by deleting k number of the least controllable and observable states. In this way, the method produces the reduced model that contains the most significant dynamics of the structure with respect to the measurements and the applied forces, as defined by the matrices B and C.

The relative error in this type of model reduction has been defined by [12, Moore 1981]:

$$\text{Relative error} = \frac{\sqrt{\sum_{i=2n-k+1}^{2n} \sigma_i^2}}{\sqrt{\sum_{i=1}^{2n-k} \sigma_i^2}} \tag{12}$$

It provides a quantitative measure of error introduced by the reduced model in calculating the response of the system.

REDUCED MODEL IN PHYSICAL COORDINATES

A problem that has been rarely addressed in the model reduction is the physical interpretation of the reduced model in conjunction with the original model. Apparently the reduced state vector $\hat{\mathbf{x}}_r$ in the balanced representation bears no obvious connection with the physical position vector $\mathbf{q}$ of Eq. (1). In fact, it may, in theory, result in all the position states being deleted, leaving only velocity states. But for structural control and measurement applications, it is desirable to provide the designer with a clear, physical relationship between the original position vector $\mathbf{q}$ and the reduced state vector $\hat{\mathbf{x}}_r$.

Such a relationship is attained by using the fact that the balanced states are linear combinations of the original states. Symbolically this is written as:

$$\begin{aligned}
\hat{x}_1 &= \sum_{j=1}^{2n} c_{1j}\, x_j \\
&\vdots \\
\hat{x}_{2n-k} &= \sum_{j=1}^{2n} c_{(2n-k)j}\, x_j \\
\hat{x}_{2n-(k-1)} &= \sum_{j=1}^{2n} c_{(2n-k+1)j}\, x_j \to 0 \\
&\vdots \\
\hat{x}_{2n} &= \sum_{j=1}^{2n} c_{2nj}\, x_j \to 0
\end{aligned} \tag{15}$$

where c_{ij}'s are the coefficients of linear combinations of $\{x_1, x_2, \ldots, x_{2n}\}$. Here the last k states are set to zero because they represent the least significant states in the balanced system [12, Moore 1981]. That is, the response with the given input and output configuration is least affected by these states. Setting each of these summations equal to zero is equivalent to imposing k constraints on the original 2n states, $\{x_1, x_2, \ldots, x_{2n}\}$. Thus, the k states among the original 2n states can be removed, with model reduction error, by the k constraints resulting from the reduction. In other words, one can construct a reduced-order model by selecting (2n-k) states out of the original 2n states. If the (2n-k) selected states from the original system are denoted by $\mathbf{x}_r = [x_{j_1} \ x_{j_2} \ \cdots \ x_{j_{2n-k}}]^T$ and the (2n-k) states of the balanced system by $\hat{\mathbf{x}}_r = [\hat{x}_1 \ \hat{x}_2 \ \cdots \ \hat{x}_{2n-k}]^T$, then the states in $\hat{\mathbf{x}}_r$ are linear combinations of the states in $\mathbf{x}_r$. Thus there exists a new transformation matrix P_r of order (2n-k)×(2n-k) such that $\mathbf{x}_r = P_r \hat{\mathbf{x}}_r$.

The above constraints and the resulting transformation allow the designer or analyst to specify which nodes (i.e. which elements of $\mathbf{q}$) of the model to be retained in the model reduction.

Now that it is shown that some members of the original states constitute the state vector $\mathbf{x}_r$ of the reduced model, the next question is how many states and which states to select from the original states. The answer to how many states, i.e., the order of the reduced model, depends on the designer's willingness to gain a smaller sized model at the expense of accuracy. The relative error in Eq. (12), defined by the singular values of Eq. (10), indicates a trade-off between error and model size. Once the order is determined, the next task is which states to select from the original states. There is no established methodology in dealing with this problem. However, strictly from the physical considerations of a given structure, the following two observations were made. First, if we recall that a pair of states--displacement and velocity--were assigned to each degree of freedom at the node when the dynamic equation (1) was converted into the state equation (2), then selecting a certain degree of freedom at a certain node is equivalent to selecting the paired states associated with that particular degree of freedom. Therefore, the paired velocity and displacement states must be either selected or deleted together, because they signify one degree of freedom at a node in the actual structure. Another observation is that for the nodes to which actuators and/or sensors are attached, the paired states representing the degree of freedom to whose direction the actuators and/or sensors function must be selected to ensure that the reduced model is under the same input and output condition as the original physical model.

In the following it is shown that the matrix P_r consists of certain rows and columns of the original transformation matrix P, and that there is a systematic way of constructing P_r from P. First, by writing the coordinate transformation, $\mathbf{x} = P\,\hat{\mathbf{x}}$, in matrix elements

$$x_i = \sum_{j=1}^{2n} p_{ij}\,\hat{x}_j \tag{16}$$

next, by the model reduction,

$$\hat{x}_{2n-(k-1)} \rightarrow 0, \ldots, \hat{x}_{2n} \rightarrow 0 \tag{17}$$

the original states are expressed as linear combinations of the first (2n-k) balanced states $\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{2n-k}\}$. The last k columns of P thereby can be removed from the expression:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{2n} \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1(2n-k)} \\ \vdots & & \vdots \\ p_{2n1} & \cdots & p_{2n(2n-k)} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_{2n-k} \end{bmatrix} \tag{18}$$

Then by selecting $\{j_1, \ldots, j_{2n-k}\}$ rows that correspond to the rows of the selected original states, the new transformation is established between $\mathbf{x}_r$ and $\hat{\mathbf{x}}_r$

$$\begin{bmatrix} x_{j_1} \\ \vdots \\ x_{j_{2n-k}} \end{bmatrix} = \begin{bmatrix} p_{j_1 1} & \cdots & p_{j_1 2n-k} \\ \vdots & & \vdots \\ p_{j_{2n-k} 1} & \cdots & p_{j_{2n-k} 2n-k} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_{2n-k} \end{bmatrix} \tag{19}$$

$$\mathbf{x}_r \quad = \quad P_r \quad \hat{\mathbf{x}}_r$$

where p_{ij}'s are the elements of the original transformation matrix P. Finally, the reduced order system $(A_r, B_r, C_r, \mathbf{x}_r)$

$$\dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r\mathbf{u}(t)$$

$$\mathbf{y}_r(t) = C_r\mathbf{x}_r(t) \tag{20}$$

is expressed in terms of a subset $\mathbf{x}_r$ of the original state vector $\mathbf{x}$ where $A_r = P_r\,\hat{A}_r\,P_r^{-1}$, $B_r = P_r\,\hat{B}_r$, and $C_r = P_r^{-1}\,\hat{C}_r$.

In summary, the model reduction procedure described in this paper can be illustrated as follows:

$$\begin{array}{llll} & P & & \\ (A, B, C, \mathbf{x}) & \Leftrightarrow & (\hat{A}, \hat{B}, \hat{C}, \hat{\mathbf{x}}) & \ldots\text{order } 2n \\ & & \Downarrow \text{model reduction} & \\ & P_r & & \\ (A_r, B_r, C_r, \mathbf{x}_r) & \Leftrightarrow & (\hat{A}_r, \hat{B}_r, \hat{C}_r, \hat{\mathbf{x}}_r) & \ldots\text{order } 2n-k \\ \text{Original} & & \text{Balanced} & \\ \text{State Space} & & \text{System} & \end{array} \tag{21}$$

where $\mathbf{x}_r$ consists of (2n-k) elements of $\mathbf{x}$, and $\hat{\mathbf{x}}_r$ consists of (2n-k) elements of $\hat{\mathbf{x}}$. In addition, the system $(\hat{A}_r, \hat{B}_r, \hat{C}_r, \hat{\mathbf{x}}_r)$ is the balanced representation of the system $(A_r, B_r, C_r, \mathbf{x}_r)$.

The following examples illustrate the proposed model reduction method.

EXAMPLE (1)

The procedure discussed in this paper is demonstrated through the example used by [12, Moore 1981]. The system $(A, B, C, \mathbf{x})$ is given

$$A = \begin{bmatrix} 0 & 0 & 0 & -150 \\ 1 & 0 & 0 & -245 \\ 0 & 1 & 0 & -113 \\ 0 & 0 & 1 & -19 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad C = [0 \;\; 0 \;\; 0 \;\; 1]$$

with an impulse input $u = \delta(t)$ of different magnitudes applied at the states x_1 and x_2. Then the transformation matrix P is calculated to be

$$P = \begin{bmatrix} 29.090 & -4.056 & 0.553 & -0.310 \\ 14.784 & 5.449 & -0.557 & 0.426 \\ 2.323 & 2.093 & -0.030 & -0.122 \\ 0.118 & 0.131 & 0.056 & 0.007 \end{bmatrix}$$

Here let us suppose that we decide to delete x_3, so that the reduced model contains the three original states, $\{x_1, x_2, x_4\}$. The transformation P_r is readily obtained by selecting the 1st, 2nd, and 4th rows and removing the 4th column of P,

$$P_r = \begin{bmatrix} 29.090 & -4.056 & 0.553 \\ 14.784 & 5.449 & -0.557 \\ 0.118 & 0.131 & 0.056 \end{bmatrix}$$

The system matrices of the reduced model are

$$A_r = P_r \hat{A}_r P_r^{-1} = \begin{bmatrix} 0.090 & -0.290 & -135.898 \\ 0.876 & 0.398 & -264.391 \\ -0.069 & 0.274 & -16.537 \end{bmatrix}$$

$$B_r = P_r \hat{B}_r = \begin{bmatrix} 3.998 \\ 1.003 \\ 0. \end{bmatrix}$$

$$C_r = P_r^{-1} \hat{C}_r = [0 \ \ 0 \ \ 1]$$

The reduced model in the physical coordinate is thus

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = A_r \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} + B_r u \qquad y_r = C_r \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$$

By setting $u = \delta(t)$, the impulse response of each original state is plotted in comparison with the difference between the two impulse responses of the state, one by the full order model and the other by the reduced order model, as shown in Figures 1-3. The difference is obtained by subtracting the response of the state in the reduced model from that of the same state in the original system.

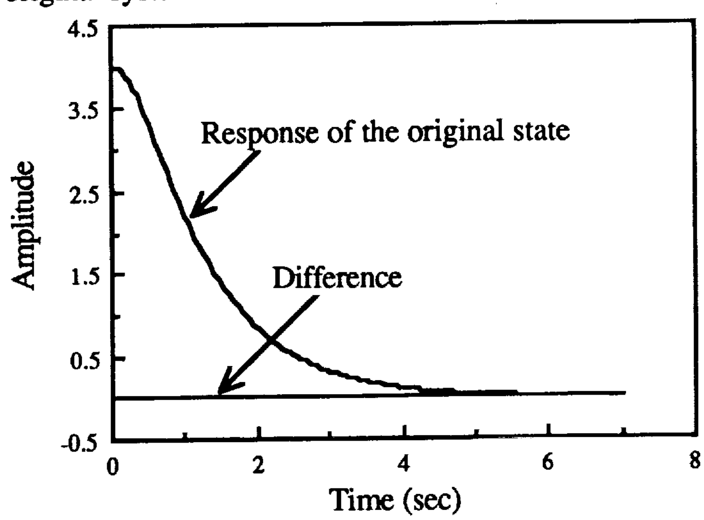

Figure 1 State #1 in Example (1)

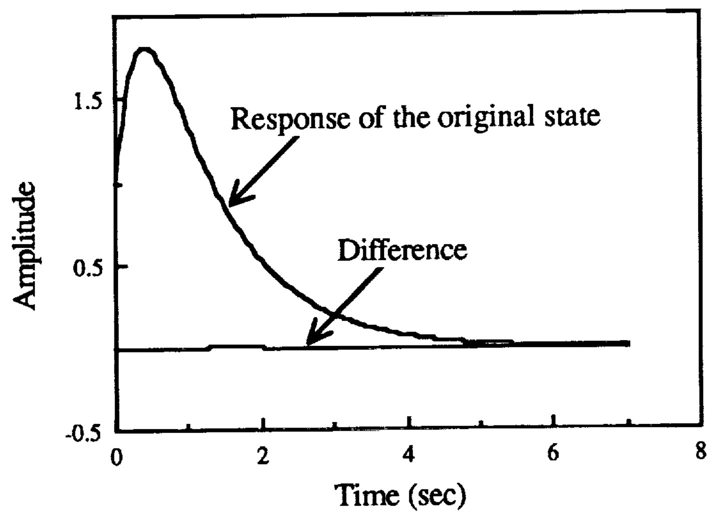

Figure 2 State #2 in Example (1)

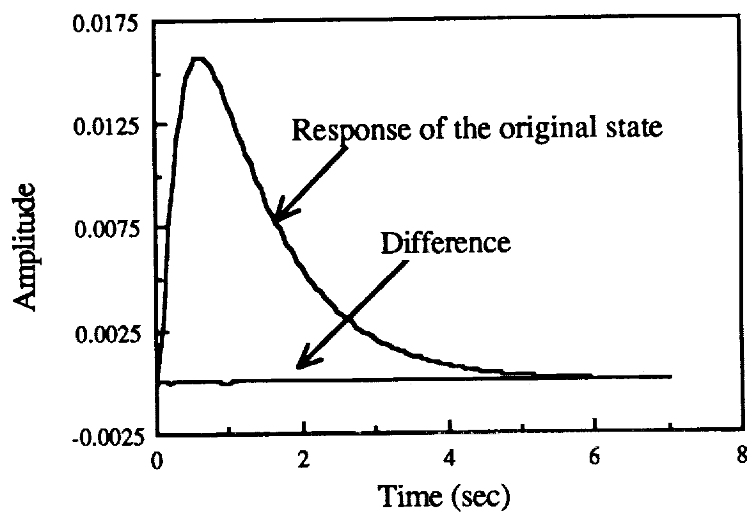

Figure 3 State #4 in Example (1)

In Figures 1-3, the difference is nearly zero, thus indicating that the reduced third-order model is indeed a respectable realization of the original fourth order system.

EXAMPLE (2)

The same procedure is applied to the following finite element model of a cantilever beam. Here the design nature of the proposed method is illustrated by assuming that actuators, machines, or sensors, will be placed at nodes 1, 4, and 5 so that they become important node points to be retained in the final model

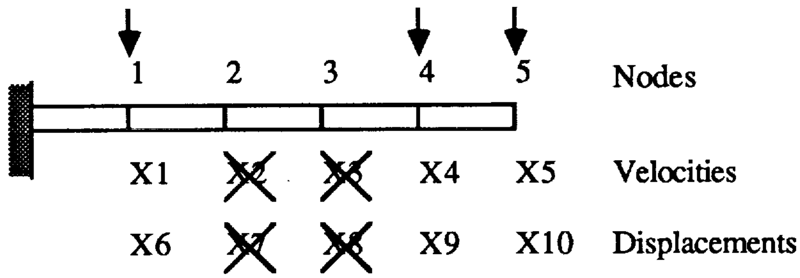

Figure 4 A cantilever beam with 5 nodes

The impulse input is applied through nodes 1, 4, and 5 in this example. Suppose that we decide to delete the states x_2, x_3, x_7, and x_8, so that the reduced model can be expressed by the remaining six states. The diagonal mass matrix is obtained by lumping mass at the node points, and the stiffness matrix by finite element analysis. The damping matrix is made up with a damping coefficient 0.002 for each mode. In Figures 4-7 the responses of the original states at Node 1 and 5 are plotted together with their differences between the responses of the states in the original system and those of the states in the reduced system. The differences are obtained in the same way as in **Example (1)**.

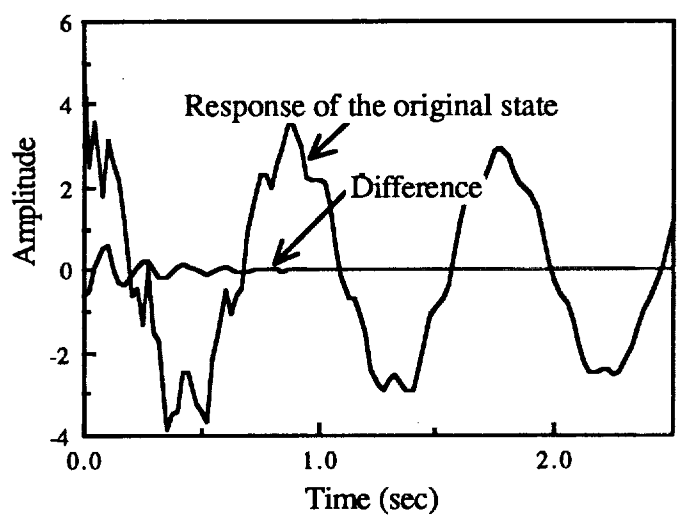

Figure 5 State #5 in Example (2): Velocity at the tip

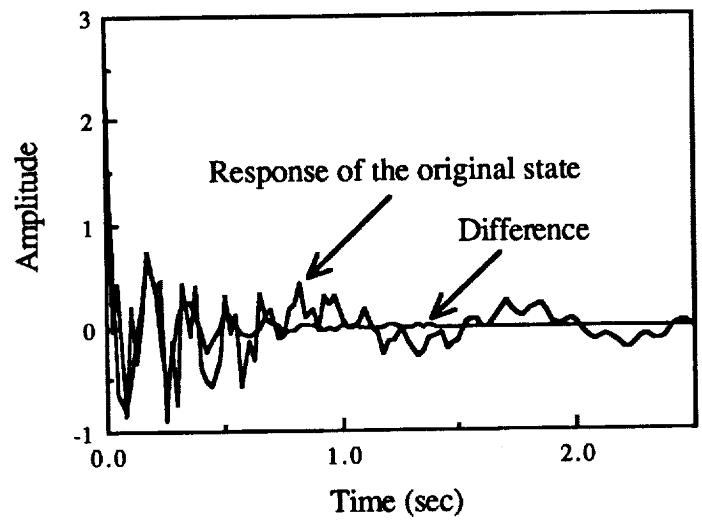

Figure 6 State #1 in Example (2): Velocity at Node #1

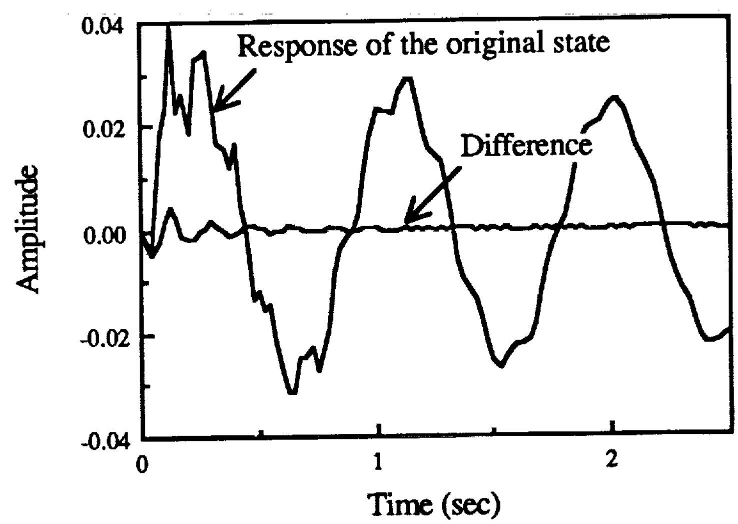

Figure 7 State #6 in Example (2): Displacement at Node #1

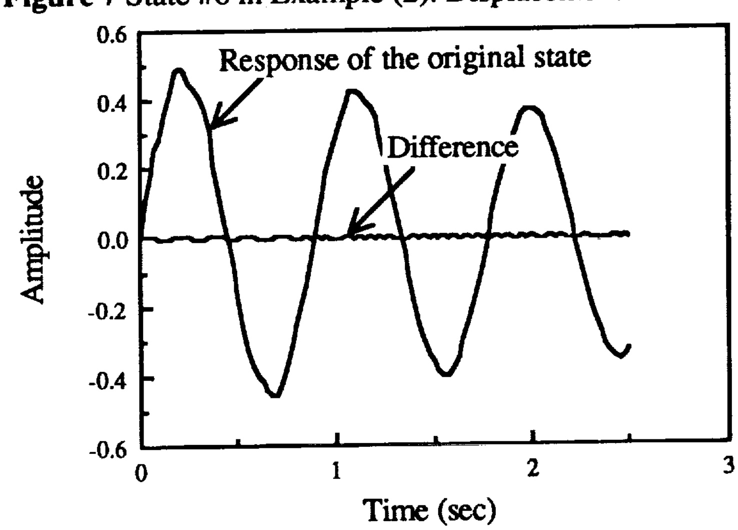

Figure 8 State #10 in Example (2): Displacement at the tip

In Figures 4-7, the difference is almost zero in comparison to the response of the original state. Some nonzero differences are detected in the transient region of the response, which indicates that the reduced model is closer to the full-order model in the steady state response region.

CONCLUSION

A model reduction method that is based on the concept of the internal balancing method is implemented along with another transformation derived from the states that are deleted during the reduction in order that the reduced model may represent the original physical model with fewer states than the original model requires.

The proposed method in this paper takes a finite element model that is reduced by Guyan's reduction, converts it into the state space form, and applies the balanced model reduction. And, through another transformation that is derived from the deleted states in reduction, the model is finally expressed by a subset of the original states. The method thereby provides a clear, physical relationship between the states in the reduced model and those in the original model. The states in the reduced model are selected directly from the original states, thus retaining the same physical meanings as in the original model. This appears to be a new and significant development in the area of model reduction. This method yields not only reduced order state space representations, but also, at the same time, reduced order transfer functions.

The application of this reduction method to a large finite element model generates a reduced model with a fewer number of nodal points, so that the analytical model improvement can be performed on the reduced model instead of a full-scale finite element model, which has been a common practice (for example, [2, Berman and Nagy 1983]). The final reduced model is of a more attractive size for dynamic simulations and subsequent structural control design.

REFERENCES

1. M. Aoki 1968 *IEEE Trans. Automat. Contr.* **AC-13**,.246-253. Control of large-scale dynamics systems by aggregation.
2. A. Berman and E.J. Nagy 1983 *AIAA Journal* **21**, 1168-1173. Improvement of a large analytical model using test data.
3. W. Garwronski and H.G. Natke 1987 *Int. J. Systems Sci.* **18**, 237-249. Balancing linear systems.
4. W. Garwronski and H.G. Natke 1986 *Int. J. Systems Sci.* **17**, 1509-1519. On balancing linear symmetric systems.
5. R. Genesio and M. Milanese 1976 *IEEE Trans. Automat. Contr.* **AC-21**, 118-122. A note on the derivation and use of reduced-order models.
6. R.J. Guyan 1965 *AIAA Journal* **3**, 380. Reduction of stiffness and mass matrices.
7. J. Hickin and N. Sinha 1980 *IEEE Trans. Automat. Contr.* **AC-25**, 1121-1127. Model reduction for linear multivariable systems.
8. D.C. Hyland and D.S. Bernstein 1985 *IEEE Trans. Automat. Contr.* **AC-30**, 1201-1211. The optimal projection equations for model reduction and the relationships among the methods of Wilson, Skelton, and Moore.
9. D.J. Inman 1989 *Vibrations with Control, Measurement and Stability*, Prentice Hall.
10. B. Irons 1965 *AIAA Journal* **3**, 961-962. Structural eigenvalue problem: elimination of unwanted variables.
11. A.J. Laub 1980 *Proc. 1980 JACC* San Francisco, CA, session FA8-E. Computation of balancing transformation
12. B.C. Moore 1981 *IEEE Trans. Automat. Contr.* **AC-26**, 17-32. Principal component analysis for linear systems: controllability, observability, and model reduction.

13. L. Pernebo and L.M. Silverman 1982 *IEEE Trans. Automat. Contr.* **AC-27**, 382-397. Model reduction via balanced state space representation.
14. S. Shokoohi, L.M. Silverman and P.M. Van Dooren 1983 *IEEE Trans. Automat. Contr.* **AC-28**, 810-822. Linear time-variable systems: balancing and model reduction.
15. R.E. Skelton and P.C. Hughes 1980 *ASME Trans. J. Dynamic Sys. Meas. Contr.* **102**, 151-158. Modal cost analysis for linear matrix-second-order systems.
16. D.A. Wilson 1970 *Proc. IEE* **117**, 1161-1165. Optimum solution of model-reduction problem.